

Generalized classical dynamical Yang-Baxter equations and moduli spaces of flat connections on surfaces

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Abstract

In this paper, we explain how generalized dynamical r -matrices can be obtained by (quasi-)Poisson reduction. New examples of Poisson structures, Poisson G -spaces and Poisson groupoid actions naturally appear in this setting. As an application, we use a generalized dynamical r -matrix induced by the gauge fixing procedure to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space of flat connections on a surface.

1 Introduction

The classical Yang-Baxter equation (CYBE) plays a key role in the theory of integrable systems. A geometric interpretation of CYBE was given by Drinfeld and gave rise to the theory of Poisson-Lie groups. The classical dynamical Yang-Baxter equation (CDYBE) is a differential equation analogue to CYBE and introduced by Felder in [8] as the consistency condition for the differential Knizhnik-Zamolodchikov-Bernard equations for correlation functions in conformal field theory on tori. It was shown by Etingof and Varchenko [7] that dynamical r -matrices correspond to Poisson Lie groupoids (a notion introduced by Weinstein [17]) in much the same way as r -matrices correspond to Poisson Lie groups. In the meantime, the classical dynamical Yang-Baxter equation is proven to be closely connected with the theory of homogeneous Poisson spaces [5], Dirac structures and Lie bialgebroids [16], see [14], [13] and references therein. Inspired by the study of Lie bialgebroids, the notion of generalized classical dynamical Yang-Baxter equation was introduced by Liu and Xu [13] in which the base manifold underlying the CDYBE can be a general Poisson manifold. Despite its importance, this subject suffered from the lack of examples for a long time.

Since Atiyah and Bott introduced canonical symplectic structures on the moduli spaces of flat connections on Riemann surface in [3], a lot of attention has been paid to the moduli spaces by mathematicians and physicists due to their rich mathematical structure and their links with a variety of topics. From the physics perspective, a major motivation for their study is their role in Chern-Simons theory. An independent mathematical motivation for

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investing moduli spaces of flat connections arises from Poisson geometry. The Atiyah-Bott symplectic structure on the moduli of flat G -connections over oriented surface Σ admits several finite dimensional descriptions. The first such description appears in Goldman's study of symplectic structures on character varieties $\text{Hom}(\pi_1(\Sigma), G)/G$, see [11]. Another possibility is to obtain the moduli space of flat G -connections on a surface $\Sigma_{g,n}$ of genus g with n punctures by (quasi-)Poisson reduction from an enlarged ambient G^{n+2g} . In the Fock-Rosly's approach [10], the Poisson structure on G^{n+2g} is described using a classical r -matrix. In the Alekseev-Malkin-Meinrenken's theory of Lie group valued moment map [2], the moduli space is obtained by a reduction of a canonical quasi-Poisson tensor on G^{n+2g} .

These two subjects of dynamical Yang-Baxter equations and moduli spaces of flat connections appear to be very different. However, some recent works indicate the possible connection between them. From the viewpoint of Hamiltonian formalism, the moduli spaces of flat connections can be viewed as constrained Hamiltonian systems. Dirac gauge fixing for the moduli space of flat $\text{ISO}(2, 1)$ -connections on a Riemann surface has been shown to give rise to generalized classical dynamical r -matrices in [15]. On the other hand, gauge fixing in the Poisson-Lie context has been shown to give rise to classical dynamical r -matrices in some cases [9].

In this paper, we deepen the connection between these two subjects by giving a systematic investigation of the theory of generalized classical dynamical r -matrices. We explain how generalized dynamical r -matrices can be obtained by (quasi-)Poisson reduction. Further, we show that new examples of Poisson structures, Poisson G -spaces and Poisson groupoid actions naturally appear in this setting. After that, we take the canonical quasi-Poisson manifold $G * G$ as an example and concretely analyze the dynamical r -matrices arising from the quasi-Poisson reduction of $G * G$. We also introduce the notion of gauge transformations for generalized dynamical r -matrices. As an application, we use a dynamical r -matrix induced by the gauge fixing procedure to give a new finite dimensional description of the symplectic structure on the moduli space. We end up with two examples, one of them was previously studied by Meusburger-Schönfeld in the framework of the $\text{ISO}(2, 1)$ -Chern-Simons theory of $(2 + 1)$ -dimensional gravity.

Our paper is structured as follows. In section 2, we recall the definition of generalized classical dynamical Yang-Baxter equations and arising some examples. After that, we give new examples of generalized dynamical r -matrices from (quasi-)Poisson reduction. We show that new Poisson structures, Poisson G -spaces and Poisson groupoid actions naturally appear in this setting. Moduli space dynamical r -matrices and gauge transformations for generalized dynamical r -matrices are introduced. In section 3, we use a dynamical r -matrix induced by the gauge fixing procedure to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space.

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2 Generalized classical dynamical Yang-Baxter equations

Let M be a manifold and G be a Lie group with the Lie algebra \mathfrak{g} . Assume $\{e_i\}_{i=1,\dots,n}$ is a basis of \mathfrak{g} . Given a tensor $\theta = \sum X_i \otimes e_i \in \Gamma(TM \otimes \mathfrak{g})$, a smooth map $r : M \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ and a linear map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$, we define the following operation

$$\delta\theta = \sum X_i \otimes \delta e_i, \quad [r, \theta] = \sum X_i \otimes [r, e_i], \quad (1)$$

$$[\theta, \theta] = \sum [X_i, X_j] \otimes e_i \wedge e_j, \quad \theta \wedge \theta = \sum X_i \wedge X_j \otimes [e_i, e_j]. \quad (2)$$

For any $\theta \in \Gamma(TM \otimes \mathfrak{g})$, we have the corresponding bundle morphism $\theta^* : T^*M \rightarrow M \times \mathfrak{g}$ and a vector bundle map $\theta^\sharp : T^*M \rightarrow \mathfrak{g}$ given by the composition of θ^* with the projection of $M \times \mathfrak{g}$ on \mathfrak{g} .

Definition 2.1. [13] *For a Poisson manifold (M, π) and a Lie algebra \mathfrak{g} , assume that there exists a tensor $\theta \in \Gamma(TM \otimes \mathfrak{g})$ such that $\theta^\sharp : (T^*M, \pi) \rightarrow \mathfrak{g}$ is a Lie algebroid morphism. A function $r \in C^\infty(M, \wedge^2 \mathfrak{g})$ is called a dynamical r -matrix coupled with the Poisson manifold (M, π) via θ if*

$$\frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^\sharp(dr), \quad (3)$$

and the generalized DYBE is satisfied:

$$\text{Alt}(\theta^* dr) + \frac{1}{2}[r, r] = \Omega, \quad (4)$$

where $\Omega \in (\wedge^3 \mathfrak{g})^\mathfrak{g}$ is an adjoint action invariant element and regarded a constant section of $\wedge^3(TM \oplus \mathfrak{g})$.

We call r a triangular dynamical r -matrix coupled with M via θ if the corresponding $\Omega = 0$ in (4). Throughout this paper, we will also use the triple (π_U, θ, r) to denote a generalized dynamical r -matrix.

Example 2.2. *Let $M = \eta^*$ where η is a Lie subalgebra of \mathfrak{g} . Then M is naturally a linear Poisson manifold. Let $\theta^\sharp : T^*\eta^* \rightarrow \mathfrak{g}$ be the natural projection: $(\xi, v) \rightarrow v$, for all $(\xi, v) \in \eta^* \times \eta$. Obviously, θ is a Lie algebroid morphism. Choose a basis of η , $\{e_1, \dots, e_k\}$,*

and let $\{x_1, \dots, x_k\}$ be the corresponding coordinate on η^* . Then $\theta = \sum_{i=1}^k \frac{\partial}{\partial x_i} \otimes e_i$. It is obvious that $[\theta, \theta] = 0$. Therefore, the condition $\frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^*(dr)$ takes the form $[r, \theta] = \pi^*(dr)$, which says that the map $r : \eta^* \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is H -equivariant, where H is the simple connected Lie group with Lie algebra η . In this case, equation $\text{Alt}(\theta^* dr) + \frac{1}{2}[r, r] = \Omega \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ turns to be the classical dynamical Yang-Baxter equation and a solution r is a classical dynamical r -matrix [8].

Similarly, we introduce a notion of the generalized Poisson-Lie dynamical Yang-Baxter equation.

Definition 2.3. For a Poisson manifold (M, π) and a Lie bialgebra (\mathfrak{g}, δ) , assume that there exists a tensor $\theta \in \Gamma(TM \otimes \mathfrak{g})$ such that $\theta^\sharp : (T^*M, \pi) \rightarrow \mathfrak{g}$ is a Lie algebroid morphism. A function $r \in C^\infty(M, \wedge^2 \mathfrak{g})$ is called a Poisson-Lie dynamical r -matrix coupled with the Poisson manifold (M, π) via θ if

- (a) $\delta\theta + \frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^\sharp(dr)$,
- (b) the generalized Poisson-Lie DYBE is satisfied:

$$\text{Alt}(\theta^* dr) + \frac{1}{2}[r, r] + \delta r = \Omega \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}. \quad (5)$$

Example 2.4. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra, G and G^* be the simply connect Lie group corresponding to \mathfrak{g} and \mathfrak{g}^* respectively. There is a bivector field π such that (G^*, π) is a Poisson Lie group. A natural section θ of $TG^* \otimes \mathfrak{g}$ is induced by the isomorphism $T^*G^* \rightarrow G^* \times \mathfrak{g}$. It is easy to check that $\theta^\sharp : T^*G^* \rightarrow \mathfrak{g}$ is a Lie algebroid morphism. A direct calculation shows that equation (a) and (b) in Definition 2.3 can be written as

$$\begin{aligned} \text{dress}_a^L(r) + [a \otimes 1 + 1 \otimes a, r] &= 0, \\ [r, r] + \text{Alt}(d^L r) + \text{Alt}((\delta \otimes \text{id})(r)) &= \Omega, \end{aligned}$$

for $r : G^* \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ and any $a \in \mathfrak{g}$, where dress_a^L denotes the left dressing vector field generalized by a and $d^L r(g) = e_i \otimes \frac{d}{dt}|_{t=0} r(\exp(-te_i) \cdot g)$ for any $g \in G^*$ and a basis $\{e_i\}$ of \mathfrak{g} . Thus a map $r : G^* \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a generalized dynamical r -matrix coupled with G^* via θ if r is a Poisson-Lie dynamical r -matrix [6].

2.1 Generalized classical dynamical r -matrices from (quasi-)Poisson reduction

In this subsection, we show that generalized classical dynamical r -matrices naturally appear in the theory of (quasi-)Poisson reduction. First, let us recall the definition of quasi-Poisson G -manifolds.

We assume that $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a quadratic Lie algebra, ϕ is the Cartan 3-tensor. In terms of an orthogonal basis $\{e_a\}$ of \mathfrak{g} , $\phi \in \wedge^3 \mathfrak{g}$ is given by

$$\phi = \frac{1}{12} f_{abc} e_a \wedge e_b \wedge e_c, \quad (6)$$

where $f_{abc} = \langle e_a, [e_b, e_c] \rangle$ are the structure constants of \mathfrak{g} . Given a G -manifold M , the Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow TM$ can be extended to an equivariant map,

$$\rho : \wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet TM,$$

preserving wedge products and Schouten brackets.

Definition 2.5. [1] *A quasi-Poisson manifold is a G -manifold M , equipped with an invariant bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that*

$$[\pi, \pi] = \rho(\phi). \quad (7)$$

Example 2.6. *Let G be a Lie group and $\{e_a\}_{a \in I}$ be an orthogonal basis of its Lie algebra \mathfrak{g} . Define a bivector on G by $\pi_G = \frac{1}{2} \sum_{a \in I} R_a \wedge L_a$, where R_a and L_a are right and left invariant vectors generated by e_a . Then (G, π_G) is a quasi-Poisson G -manifold, where G acts on itself by conjugation.*

Generally, the G -invariant functions on a quasi-Poisson manifold (M, π) is a Poisson algebra under the binary bracket induced by π . Thus we obtain a Poisson algebra on $C^\infty(M)^G$.

Let M be a G -manifold with G acting locally freely and $\rho : \mathfrak{g} \rightarrow TM$ the corresponding infinitesimal action. We use the same symbol ρ to denote the following natural extension map:

$$\rho : \wedge^\bullet (TM \oplus \mathfrak{g}) \rightarrow \wedge^\bullet TM. \quad (8)$$

Throughout this paper, we denote the anti-symmetrization of any section $A \in \Gamma(\wedge^\bullet (TM) \otimes \wedge^\bullet \mathfrak{g})$ by $\hat{A} \in \Gamma(\wedge^\bullet (TM \oplus \mathfrak{g}))$. Thus, if $\theta = f^{ia}(x) \frac{\partial}{\partial x_i} \otimes e_a \in \Gamma(TM \otimes \mathfrak{g})$ in local coordinates $\{x_i\}$ of U and a basis $\{e_a\}$ of \mathfrak{g} , we have $\hat{\theta} = f^{ia}(x) \frac{\partial}{\partial x_i} \wedge e_a$.

Theorem 2.7. *Let $U \subset M$ be a cross-section of the G action and π_M be a bivector field on M . Then there exists a unique triple (π_U, θ, r) , where $\pi_U \in \Gamma(\wedge^2 TU)$, $\theta \in \Gamma(TU \otimes \mathfrak{g})$ and $r : U \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ such that*

$$\pi_M|_U = \pi_U - \rho(\hat{\theta})|_U + \rho(r)|_U. \quad (9)$$

Moreover,

(a) *if π_M is a G -invariant Poisson tensor on M , then (U, π_U) is a Poisson manifold and r is a triangular dynamical r -matrix coupled with U via θ .*

(b) *if π_M is a quasi-Poisson tensor on M , then (U, π_U) is a Poisson manifold and r is a dynamical r -matrix coupled with U via θ with respect to the Cartan 3-tensor, i.e., $\Omega = -\frac{1}{2}\phi$ in the generalized CDYBE.*

Proof. Because G acts locally freely and U is a cross-section, for any $x \in U$ there is a canonical splitting $T_x M = T_x U \oplus \rho_x(\mathfrak{g})$ of the sequence

$$0 \rightarrow \mathfrak{g} \rightarrow T_x M \rightarrow T_x U \rightarrow 0.$$

Thus, there exists unique $\pi_U \in \Gamma(\wedge^2 TU)$, $\theta \in \Gamma(TU \otimes \mathfrak{g})$ and $r : U \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ such that

$$\pi_M|_U = \pi_U - \rho(\hat{\theta})|_U + \rho(r)|_U,$$

where π_U is tangent to U .

If π_M is a G -invariant Poisson tensor or quasi-Poisson tensor, it induces a Poisson bracket $\{\cdot, \cdot\}$ on U . From the expression (9) and the fact $\rho(e)f' = 0$ for any $e \in \mathfrak{g}$ and $f' \in C^\infty(M)^G$, we have that $\{f, g\} = \pi_U(df, dg)$ for any $f, g \in C^\infty(U)$. This is to say (U, π_U) is a Poisson manifold. To complete the proof, we have to check that the required compatibility condition and the generalized CDYBE are satisfied by the triple (π_U, θ, r) . This proof is similar to the ones of Theorem 2.8 and Theorem 2.12. ■

Theorem 2.7 suggests the following construction which generalizes the construction for ordinary classical dynamical r -matrices in [18]. Given a manifold M , $M \times G$ carries natural right and left G -actions defined respectively by $(x, p) \cdot g = (x, pg)$ and $g \cdot (x, p) = (x, gp)$ for all $x \in M$, $p, g \in G$. Let ρ^L and ρ^R denote respectively the infinitesimal left and right G -actions. Then we have:

Theorem 2.8. *Let (M, π_M) be a Poisson manifold and $\theta \in \Gamma(TM \otimes \mathfrak{g})$. Then any smooth function $r : M \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ induces a left G -invariant bivector π_r on $M \times G$ which is given by*

$$\pi_r = \pi_M + \rho^L(\hat{\theta}) + \rho^L(r), \quad (10)$$

and

- (a) π_r is a Poisson tensor iff r is a triangular generalized dynamical r -matrix.
- (b) π_r is a quasi-Poisson tensor iff r is a generalized dynamical r -matrix with respect to the Cartan 3-tensor.

Proof. Note that the vector field on $M \times G$ has a natural bigrading: elements in TM have degree $(1, 0)$ while elements in TG have degree $(0, 1)$. It is simple to see that $[\pi_M, \pi_M]$ is of degree $(3, 0)$, $[\pi_M, \rho^L(\hat{\theta})]$ is of degree $(2, 1)$, $[\pi_M, \rho^L(r)]$ is of degree $(1, 2)$ and $[\rho^L(r), \rho^L(r)]$ is of degree $(0, 3)$. On the other hand, $[\rho^L(\hat{\theta}), \rho^L(r)]$ consists of elements of degree $(1, 2)$ and of $(0, 3)$ and $[\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]$ consists of elements of degree $(2, 1)$ and of $(1, 2)$. For any $S \in \wedge^3(TM \oplus TG)$, let $S = \sum_{0 \leq i, j \leq 3} S^{(i, j)}$ be its decomposition with respect to this bigrading. The following equations can be verified by a direct computation:

$$[\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]^{(1, 2)} = \rho^L(\widehat{[\theta, \theta]}), \quad [\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]^{(2, 1)} = 2\rho^L(\widehat{\theta \wedge \theta}), \quad (11)$$

$$[\rho^L(\hat{\theta}), \rho^L(r)]^{(0, 3)} = \rho^L(Alt(\theta^* dr)), \quad [\pi_M, \rho^L(r)] = \rho^L(\pi_M^\sharp(dr)), \quad (12)$$

$$[\rho^L(\hat{\theta}), \rho^L(r)]^{(1, 2)} = -\rho^L(\widehat{[r, \theta]}), \quad [\pi_M, \rho^L(\hat{\theta})] = \rho^L(d_{\pi_M} \theta) \quad (13)$$

where the operations $[\theta, \theta]$, $\theta \wedge \theta$ and $[r, \theta]$ for $\theta \in \Gamma(TM \otimes \mathfrak{g})$ and $r \in C^\infty(M, \wedge^2 \mathfrak{g})$ are defined as (1) and (2). Eventually, by using the facts $[\rho^L(r), \rho^L(r)] = \rho^L([r, r])$ and $[\pi_M, \pi_M] = 0$ we have

$$\begin{aligned} [\pi_r, \pi_r] &= [\pi_M + \rho^L(\hat{\theta}) + \rho^L(r), \pi_M + \rho^L(\hat{\theta}) + \rho^L(r)] \\ &= [\rho^L(\hat{\theta}), \rho^L(\hat{\theta})] + 2[\pi_M, \rho^L(\hat{\theta})] + 2[\rho^L(\hat{\theta}), \rho^L(r)] + 2[\pi_M, \rho^L(r)] + [\rho^L(r), \rho^L(r)] \\ &= 2\rho^L(Alt(\theta^* dr) + \frac{1}{2}[r, r]) + 2\rho^L([\widehat{\theta, \theta}] - [\widehat{r, \theta}] + \widehat{\pi^*(dr)}) + 2\rho^L(\widehat{\theta \wedge \theta} + d_{\pi_M} \theta). \end{aligned}$$

Note that the map $\theta^\sharp : T^*M \rightarrow \mathfrak{g}$ is a Lie algebroid morphism is equivalent to

$$\widehat{\theta \wedge \theta} + d_{\pi_M} \theta = 0.$$

Therefore we have that $[\pi_r, \pi_r] = 0$ iff r is a triangular generalized dynamical r -matrix and $[\pi_r, \pi_r] = \rho^R(\phi)$ iff r is a generalized dynamical r -matrix with respect to $\Omega = -\frac{1}{2}\phi$. ■

Similarly, we have the following theorem.

Theorem 2.9. *Let (N, π_N) be a quasi-Poisson G -space and $\rho : \mathfrak{g} \rightarrow TN$ be the infinitesimal action. Then for any generalized dynamical r -matrix coupled with (M, π_M) via θ with respect to $\Omega = -\frac{1}{2}\phi$,*

$$\pi := \pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r) \tag{14}$$

is a Poisson tensor on $M \times N$.

Proof. We need to prove $[\pi, \pi] = 0$. Note that $[\rho(r), \pi_N] = [\rho(\hat{\theta}), \pi_N] = 0$ because of the invariance of π_N . Thus we have

$$\begin{aligned} [\pi, \pi] &= [\pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r), \pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r)] \\ &= [\pi_M, \pi_M] + [\pi_N, \pi_N] + \rho([r, r]) + 2[\rho(\hat{\theta}), \rho(r)] + 2[\pi_M, \rho(\hat{\theta})] + 2[\pi_M, r] \\ &= [\pi_N, \pi_N] + 2\rho(Alt(\theta^* dr) + \frac{1}{2}[r, r]) \\ &= \rho(\phi) - \rho(\phi) = 0. \end{aligned}$$

This finishes the proof. ■

Now we discuss the relation between the generalized dynamical r -matrix and the reduction of the fusion product of two quasi-Poisson manifolds. Let M, N be two G -manifolds and ρ_M, ρ_N be the corresponding infinitesimal G action. We define a bivector field on $M \times N$ by

$$\Phi = \sum_{a \in I} \rho_M(e_a) \wedge \rho_N(e_a),$$

where $\{e_a\}_{a \in I}$ is an orthogonal basis of \mathfrak{g} .

Proposition 2.10. [1] *If (M, π_M) and (N, π_N) are two quasi-Poisson G -manifolds, then $\pi_M + \pi_N - \Phi$ gives a quasi-Poisson structure on $M \times N$ for the diagonal G -action. This quasi-Poisson manifold, denoted by $M * N$, is called the fusion product of M and N .*

Example 2.11. *Let (G, π_G) be the quasi-Poisson G -manifold given in Example 2.6. By using the fusion product between G with itself, we get a quasi-Poisson manifold $G * G$. Let R_a^i and L_a^i denote the right and left invariant vector fields on i -th copy of $G \times G$ generated by e_a , then the quasi-Poisson tensor takes the form:*

$$\pi_{G^2} = \frac{1}{2} \sum_a (R_a^1 \wedge L_a^1 + R_a^2 \wedge L_a^2 + (L_a^1 - R_a^1) \wedge (L_a^2 - R_a^2)). \quad (15)$$

Let (M, π_M) and (N, π_N) be two quasi-Poisson G -manifolds. We assume G acts locally freely on M . Let $U \subset M$ be any cross-section of the G action on M . By Theorem 2.7, associated to U , there is a triple (π_U, θ, r) such that r is a generalized dynamical r -matrix coupled with (U, π_U) via θ . On the other hand, $U \times N$ is a cross-section of the diagonal G action on $M \times N$. So it inherits a Poisson structure π_{red} from the reduction of the quasi-Poisson structure on $M * N$.

Theorem 2.12. *The Poisson tensor π_{red} on $U \times N$ takes the form*

$$\pi_{\text{red}} = \pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N. \quad (16)$$

Before giving a proof, we show the following lemma which says that in the reduction level, fusion product and direct product are same.

Lemma 2.13. *Let M and N be two quasi-Poisson G -manifolds. Then for any diagonal G -invariant functions $f, g \in C^\infty(M \times N)^G$, $\Phi(df, dg) = 0$. Moreover, $\pi_M + \pi_N$ induces a Poisson algebra structure on $C^\infty(M \times N)^G$ which is same as the Poisson algebra on $C^\infty(M * N)^G$.*

Proof. Note that $\Phi = \sum \rho_M(e_a) \wedge \rho_N(e_a)$, and $(\rho_M(e_a) + \rho_N(e_a))f = 0$ for f G -invariant function on $M \times N$. So $\Phi = -\sum \rho_N(e_a) \wedge \rho_N(e_a) = 0$ when restricts to G -invariant functions. ■

Proof of Theorem 2.12 For any $f, g \in C^\infty(U \times N)$, let $f', g' \in C^\infty(M \times N)^G$ be the diagonal G -invariant extension of f, g respectively. Then by Lemma 2.13, $\pi_{\text{red}}(df, dg) = (\pi_M + \pi_N)(df', dg')|_{U \times N}$. Following Theorem 2.7, we have

$$\pi_M|_U = \pi_U - \rho_M(\hat{\theta})|_U + \rho_M(r)|_U.$$

Together with the fact $(\rho_M(e_a) + \rho_N(e_a))F = 0$ for any $F \in C^\infty(M \times N)^G$, we get

$$(\pi_M + \pi_N)(df', dg')|_{U \times N} = (\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df', dg')|_{U \times N}.$$

Note that $\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N$ is tangent to $U \times N$ and $f'|_U = f, g'|_U = g$, therefore,

$$(\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df', dg')|_{U \times N} = (\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df, dg).$$

This is to say $\pi_{\text{red}} = \pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N$. ■

2.2 Classical dynamical r -matrices and Poisson G -spaces

Let \mathfrak{g} be a complex simple Lie algebra, $\kappa \in (S^2\mathfrak{g})^{\mathfrak{g}}$ the element corresponding to the Killing form on \mathfrak{g} and $\Lambda = \frac{\varepsilon}{2}\kappa + r_0$ any classical quasi-triangular r -matrix with skew-symmetric part r_0 . Thus we have

$$\pi_G := \rho^L(r_0) - \rho^R(r_0), \quad (17)$$

where ρ^R and ρ^L denote respectively the infinitesimal actions of the right and left translations, defines a Lie Poisson structure on G .

Let H be a Lie subgroup of G with the Lie algebra \mathfrak{h} and $r = \frac{\varepsilon}{2}\kappa + A_r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ a classical dynamical r -matrix with the skew-symmetric part A_r , i.e., r is a H -equivariant map and satisfies classical dynamical Yang-Baxter equation

$$\text{Alt}(\theta^*(dA_r)) + \frac{1}{2}[A_r, A_r] = \frac{\varepsilon^2}{4}[\kappa^{12}, \kappa^{23}] = -\frac{\varepsilon^2}{4}\phi, \quad (18)$$

where ϕ is the Cartan 3-tensor, $\theta = \frac{\partial}{\partial x^i} \wedge e_i$ for a basis $\{e_i\}$ of \mathfrak{h} and the corresponding coordinates $\{x^i\}$ on \mathfrak{h}^* .

On the other hand, $\rho^R(r_0)$ is a "quasi-Poisson-like" bivector(not G -invariant) on G with respect to the left G -action, i.e., $[\rho^R(r_0), \rho^R(r_0)] = -\rho^R[r_0, r_0] = \frac{\varepsilon^2}{4}\rho^R(\phi) = \frac{\varepsilon^2}{4}\rho^L(\phi)$. Therefore, similar to Theorem 2.9, we have

Proposition 2.14. *Associated to the dynamical r -matrix A_r , there is a Poisson structure π_r on $\mathfrak{h}^* \times G$ defined by*

$$\pi_r = \pi_{KKS} + \rho^L(\theta) + \rho^L(A_r) - \rho^R(r_0), \quad (19)$$

where π_{KKS} is the Kirillov-Kostant-Souriau (KKS) Poisson structure on \mathfrak{h}^* .

Proof. Note that the Schouten brackets $[\pi_{KKS}, \rho^R(r_0)]$, $[\rho^L(\theta), \rho^R(r_0)]$ and $[\rho^L(A_r), \rho^R(r_0)]$ are zero. Similar to the proof of Theorem 2.9, a straightforward calculation shows that $[\pi_r, \pi_r] = 0$. ■

Now let us consider the H action on $\mathfrak{h}^* \times G$ given by

$$h \cdot (x, g) = (Ad_h^*x, gh), \quad \forall h \in H, x \in \mathfrak{h}^*, g \in G. \quad (20)$$

Proposition 2.15. *The Poisson tensor π_r on $\mathfrak{h}^* \times G$ is H -invariant.*

Proof. In the expression of the Poisson tensor π_r on $\mathfrak{h}^* \times G$, the components π_{KKS} , $\rho^L(\theta)$ and $\rho^R(r_0)$ are obviously H -invariant. Therefore, π_r is H -invariant as long as $\rho^L(A_r)$ is H -invariant which can be seen from the fact that the map $A_r : \mathfrak{h}^* \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is H -equivariant. ■

Recall that an action of the Poisson Lie group (G, π_G) on a Poisson manifold M is said to be Poisson if the action map $G \times M \rightarrow M$ is a Poisson map, where $G \times M$ is equipped

with the product Poisson structure. In our case, $\mathfrak{h}^* \times G$ carries a natural left G -action, i.e.,

$$g_1 \cdot (x, g_2) = (x, g_1 g_2), \quad \forall x \in \mathfrak{h}^*, g_1, g_2 \in G. \quad (21)$$

Proposition 2.16. *$(\mathfrak{h}^* \times G, \pi_r)$ is a Poisson (G, π_G) -space with respect to the left G -action.*

Proof. By definition, $G \times (\mathfrak{h}^* \times G) \rightarrow \mathfrak{h}^* \times G$ is a Poisson map if and only if for all $g_1 \in G$ and $a = (x, g_2) \in \mathfrak{h}^* \times G$,

$$\pi_r(g_1 \cdot a) = g_{1*}(\pi_r(a)) + a_*(\pi_G(g_1)), \quad (22)$$

where a in the last term denotes the map $a : G \rightarrow \mathfrak{h}^* \times G$, $a(g_1) := (x, g_1 g_2)$, for any $g_1 \in G$. Equation (22) can be obtained by the following calculation

$$\begin{aligned} \pi_r(g_1 \cdot a) &= \pi_r(x, g_1 g_2) \\ &= \pi_{KKS}(x, g_1 g_2) + L_{g_1 g_2} \theta(x) + L_{g_1 g_2} A_r(x) - R_{g_1 g_2} r_0 \\ &= L_{g_1}(\pi_{KKS}(x, g_2) + \theta(x, g_2)) + L_{g_1}(L_{g_2} A_r(x) - R_{g_2} r_0) + R_{g_2}(L_{g_1} r_0 - R_{g_1} r_0) \\ &= g_{1*}(\pi_r(a)) + a_*(\pi_G(g_1)), \end{aligned} \quad (23)$$

where $R_g(L_g)$ denotes the right(left) translation from the identity element to g . ■

By Poisson reduction, the H -invariant Poisson tensor π_r on $\mathfrak{h}^* \times G$ induces a Poisson structure π_s on $(\mathfrak{h}^* \times G)/H = G \times_H \mathfrak{h}^*$, where $(G, G/H)$ is a principal H -bundle and $G \times_H \mathfrak{h}^*$ is regarded as the associated vector bundle over G/H with respect to the coadjoint representation. Since the G -action and the H -action on $\mathfrak{h}^* \times G$ commute, $G \times_H \mathfrak{h}^*$ carries an induced left G action, $G \times (G \times_H \mathfrak{h}^*) \rightarrow G \times_H \mathfrak{h}^*$. Furthermore, we have

Theorem 2.17. *$(G \times_H \mathfrak{h}^*, \pi_s)$ is a Poisson (G, π_G) -space.*

The above theorem is a consequence of the following general proposition.

Proposition 2.18. *Let (M, π_M) be a Poisson (G, π_G) -space. Suppose a Lie group H acts on M freely, commuting with the G action such that π_M is H -invariant. Then the reduced Poisson manifold $(M/H, \pi_{red})$ is a Poisson (G, π_G) -space with respect to the induced G -action.*

Proof. By definition, for any $g \in G$ and $a \in \mathfrak{h}^* \times G$, we have

$$\pi_r(g \cdot a) = g_* \pi_r(a) + a_*(\pi_G(g)). \quad (24)$$

Quotienting the H action on the two sides, this says that the left action map of G on $(\mathfrak{h}^* \times G)/H$ is a Poisson map. ■

The construction in this subsection is inspired by Lu's work [14]. Our discussion gives an interpretation of her construction of Poisson homogeneous spaces from dynamical r -matrices.

2.3 Generalized dynamical r -matrices from the reduction of $(G \times G, \pi_{G^2})$

Let $(G \times G, \pi_{G^2})$ be the quasi-Poisson manifold given in Example 2.11. For a conjugacy class \mathcal{C} in G , identify the tangent space $T_g \mathcal{C}$ at g with \mathfrak{g}_g^\perp , where \mathfrak{g}_g is the Lie algebra of the stabilizer of g . The operator $\text{Ad}_g - 1$ is invertible on \mathfrak{g}_g^\perp . Thus we get a linear operator

$$\frac{\text{Ad}_g + 1}{\text{Ad}_g - 1} |_{\mathfrak{g}_g^\perp} := \left(\frac{\text{Ad}_g + 1}{\text{Ad}_g - 1} \right) \circ \text{Pr}_{\mathfrak{g}_g^\perp} : \mathfrak{g} \rightarrow \mathfrak{g},$$

where $\text{Pr}_{\mathfrak{g}_g^\perp}$ is the projection of \mathfrak{g} on \mathfrak{g}_g^\perp .

Proposition 2.19 (Proposition 3.4, [1]). $\sum_{a \in I} R_a \wedge L_a = \sum_{a, b \in I} \frac{1}{2} \left(\frac{\text{Ad}_g + 1}{\text{Ad}_g - 1} |_{\mathfrak{g}_g^\perp} \right)_{ab} X_a \wedge X_b$ as bivector fields on G , where $\{e_a\}_{a \in I}$ is a basis of \mathfrak{g} and $X_a = L_a - R_a$ for any $e_a \in \mathfrak{g}$.

As a result, for any conjugacy classes $\mathcal{C}_1, \mathcal{C}_2$ in G , $\pi_{G^2}|_{\mathcal{C}_1 \times \mathcal{C}_2}$ is tangent to $\mathcal{C}_1 \times \mathcal{C}_2$, i.e., $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_{G^2}|_{\mathcal{C}_1 \times \mathcal{C}_2})$ is a quasi-Poisson manifold.

Now let $T \subset G$ be a maximal torus, $p \in \mathcal{C}_1 \cap T$ and G_p its isotropic group under the conjugation action of G , \mathfrak{g}_p the Lie algebra of G_p . For any open set in \mathcal{C}_2 where the conjugation G_p action is locally free, we can choose a cross-section U of the G_p action. Thus, we have $\{p\} \times U$ is a cross-section of the simultaneous conjugation G action on $\mathcal{C}_1 \times \mathcal{C}_2$. Following Theorem 2.7, associated to $\{p\} \times U$, there is a dynamical r -matrix $(\pi_{p \times U}, \theta, r)$. To write down it explicitly, we introduce a function $H \in C^\infty(U, \text{End}(\mathfrak{g}))$ in the following way. For any point $x \in U$, let \mathfrak{g}'_x be the subspace of \mathfrak{g} defined by

$$\mathfrak{g}'_x = \left\{ u \in \mathfrak{g} \mid \frac{d}{dt} \Big|_{t=0} \exp(tu) \cdot x \cdot \exp(-tu) \in T_x U \right\}.$$

Because U is a cross-section of the conjugation G_p action, we have a set of direct product decompositions of \mathfrak{g} as $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{g}'_x$ parameterized by the coordinates on U . Then we define $H_x \in \text{End}(\mathfrak{g})$ to be the projection of \mathfrak{g} on \mathfrak{g}'_x . Thus associated to any $e_a \in \mathfrak{g}$, there is a function $H(e_a) \in C^\infty(U, \mathfrak{g})$ and a vector field $H(X_a) \in \Gamma(TU)$ defined by

$$H(X_a)|_x := \frac{d}{dt} \Big|_{t=0} \exp(tH_x(e_a)) \cdot x \cdot \exp(-tH_x(e_a)), \quad \forall x \in U. \quad (25)$$

Theorem 2.20. *The triple $(\pi_{p \times U}, \theta, r)$ from the reduction of the quasi-Poisson tensor π_{G^2}*

on $\mathcal{C}_1 \times \mathcal{C}_2$ is given by

$$\pi_{p \times U} = \frac{1}{2} \sum_{a,b} \left(\left(\frac{\text{Ad}_p + 1}{\text{Ad}_p - 1} \Big|_{\mathfrak{g}_p^\perp} \right)_{ab} + \frac{\text{Ad}_x + 1}{\text{Ad}_x - 1} \Big|_{\mathfrak{g}_x^\perp} \right)_{ab} H(X_a) \wedge H(X_b), \quad (26)$$

$$\begin{aligned} \theta &= \frac{1}{2} \sum_a H(X_a) \otimes e_a + \frac{1}{2} \left(\frac{\text{Ad}_p + 1}{\text{Ad}_p - 1} \Big|_{\mathfrak{g}_p^\perp} \right)_{ab} H(X_a) \otimes H(e_b) + \\ &\quad \frac{1}{2} \sum_{a,b} \left(\frac{\text{Ad}_x + 1}{\text{Ad}_x - 1} \Big|_{\mathfrak{g}_x^\perp} \right)_{ab} H(X_a) \otimes (H(e_b) - e_b), \end{aligned} \quad (27)$$

$$\begin{aligned} r &= \frac{1}{2} \sum_a e_a \wedge H(e_a) + \frac{1}{2} \left(\frac{\text{Ad}_p + 1}{\text{Ad}_p - 1} \Big|_{\mathfrak{g}_p^\perp} \right)_{ab} H(e_a) \wedge H(e_b) + \\ &\quad \frac{1}{2} \sum_{a,b} \left(\frac{\text{Ad}_x + 1}{\text{Ad}_x - 1} \Big|_{\mathfrak{g}_x^\perp} \right)_{ab} (e_a - H(e_a)) \wedge (e_b - H(e_b)). \end{aligned} \quad (28)$$

Proof. Note that for the cross-section $p \times U$ of the G_p action on $\mathcal{C}_1 \times \mathcal{C}_2$, the triple $(\pi_{p \times U}, \theta, r)$ corresponds to the decomposition of $\pi_{G^2}|_{p \times U}$ with respect to TU and the complement $\rho(\mathfrak{g})|_U$ generated by the diagonal G -action, where $\rho(e_a) = X_a^1 + X_a^2$ for any $e_a \in \mathfrak{g}$. On the other hand, from Example 2.11 and Proposition 2.19, we have at any point $(p, x) \in \{p\} \times U$,

$$\pi_{G^2} = \frac{1}{4} \sum_{a,b} \left(\frac{\text{Ad}_p + 1}{\text{Ad}_p - 1} \Big|_{\mathfrak{g}_p^\perp} \right)_{ab} X_a^1 \wedge X_b^1 + \frac{1}{4} \sum_{a,b} \left(\frac{\text{Ad}_x + 1}{\text{Ad}_x - 1} \Big|_{\mathfrak{g}_x^\perp} \right)_{ab} X_a^2 \wedge X_b^2 + \sum_a X_a^1 \wedge X_a^2. \quad (29)$$

With the help of this expression, we just need to compute the decomposition of $X_a^i \in \Gamma(T(\mathcal{C}_1 \times \mathcal{C}_2)|_U)$ along the two directions TU and $\rho(\mathfrak{g})|_U$. After a direct computation, we get the following decompositions:

$$X_a^1|_{p \times U} = -H(X_a) + \rho(H(e_a))|_{p \times U}, \quad (30)$$

$$X_a^2|_{p \times U} = H(X_a) + \rho(e_a - H(e_a))|_{p \times U}, \quad (31)$$

where $\rho(H(e_a)) \in \Gamma(T(\mathcal{C}_1 \times \mathcal{C}_2))$ is given by $\rho(H(e_a))|_{y,x} = (X_{H_x(e_a)}^1 + X_{H_x(e_a)}^2)|_{y,x}$ for all $(y, x) \in \mathcal{C}_1 \times \mathcal{C}_2$. To be precise, by the definition of H , $e_a - H(e_a) \in C^\infty(U, \mathfrak{g}_p)$ where \mathfrak{g}_p is the Lie algebra of the isotropic group G_p , so we get $X_a^1 = X_{H(e_a)}^1$ when restricts to $p \times U$. Thus we have

$$-H(X_a) + \rho(H(e_a))|_{p \times U} = -H(X_a) + X_{H(e_a)}^1|_{p \times U} + H(X_a) = X_a^1|_{p \times U}.$$

A similar calculation gives the equation (31).

In the end, we get the expression of the triple $(\pi_{p \times U}, \theta, r)$ by plugging (30) and (31) in the expression of $\pi_{G^2}|_{p \times U}$. ■

We refer to the generalized dynamical r -matrices associated to two conjugacy classes in G as moduli space generalized dynamical r -matrices.

Let us take $G = \text{SU}(2)$ for a concrete example. Let

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be a basis of $\mathfrak{su}(2)$ and $\mathcal{C} \subset \text{SU}(2)$ the conjugacy class through $p = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then \mathcal{C} can be identified with the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and an element in \mathcal{C} takes the form $\begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}$. The diagonal matrix $\begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}$ acts on $\mathcal{C} = S^2 \in \mathbb{R}^3$ by rotation with respect to x -axis, i.e., $e^{i\beta} \circ (x, y, z) = (x, e^{2i\beta}y, e^{2i\beta}z)$. For this S^1 action, we choose a simple cross-section

$$U := \{(x, y, z) \mid -1 < x < 1, y = 0, z > 0\} \subset \mathcal{C} = S^2 \in \mathbb{R}^3.$$

U can be parameterized by α where $x = \sin \alpha$ and $z = \cos \alpha$. Recall that at any point $\alpha \in U$, we have the subspace of \mathfrak{g} defined by

$$\mathfrak{g}'_\alpha = \{e \in \mathfrak{g} \mid \frac{d}{dt}|_{t=0} \exp(te) \cdot \alpha \cdot \exp(-te) \in T_\alpha U\}.$$

A direct calculation gives

Proposition 2.21. *In terms of e_1, e_2, e_3 , $\mathfrak{g}'_\alpha = \text{Span}\{e_1 + \tan \alpha e_2, e_3\}$.*

It follows that the function $H \in C^\infty(U, \text{End}(\mathfrak{g}))$ is given by

$$H(e_1) = 0, \quad H(e_2) = \cot \alpha e_1 + e_2, \quad H(e_3) = e_3.$$

The corresponding vector fields on U generated by adjoint action are given by

$$H(X_1) = 0, \quad H(X_2) = 0, \quad H(X_3) = \frac{\partial}{\partial \alpha}.$$

Another straightforward computation shows that $\mathfrak{g}_\alpha^\perp = \text{Span}\{\tan \alpha e_1 - e_2, e_3\}$, where \mathfrak{g}_α is the Lie subalgebra of the stabilizer of \mathfrak{g} at $\alpha \in U$. However, we have

$$(\text{Ad}_\alpha + 1)e_3 = (\text{Ad}_\alpha + 1)(\tan \alpha e_1 - e_2) = 0.$$

It is to say that $(\frac{\text{Ad}_\alpha + 1}{\text{Ad}_\alpha - 1}|_{\mathfrak{g}_\alpha^\perp}) : \mathfrak{g}_\alpha^\perp \rightarrow \mathfrak{g}_\alpha^\perp$ is a zero transformation.

Eventually, by Theorem 2.20, the dynamical r -matrix associated to the local section $p \times U$ of $(\mathcal{C} \times \mathcal{C})/G$ is given by

$$r = \tan \alpha e_1 \wedge e_2, \quad \theta = \frac{\partial}{\partial \alpha} \otimes e_3. \quad (32)$$

Actually, $r : \eta^* \rightarrow \mathfrak{su}(2) \otimes \mathfrak{su}(2)$ is a quasi-triangular dynamical r -matrix in ordinary sense where $\eta = \text{Span}\{e_3\}$ is an abelian subalgebra of $\mathfrak{su}(2)$.

2.4 Gauge transformations of generalized classical dynamical r -matrices

Let G be a Lie group and r be a dynamical r -matrix coupled with (U, π_U) via θ with respect to Ω . Let $g : U \rightarrow G$ be a smooth map. We define a function $\langle g^*\Theta, \theta \rangle : U \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, where $\Theta = g^{-1}dg$ is the Cartan one form on G and $\langle \cdot, \cdot \rangle$ is the pairing between one form and vector field components of $g^*\Theta \in \Omega^1(U, \mathfrak{g})$ and $\theta \in \Gamma(TU \otimes \mathfrak{g})$ respectively. Similarly, with the help of the pairing between vectors and forms, we can define a function $\langle \pi_U, g^*\Theta \wedge g^*\Theta \rangle : U \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ and a section $\langle \pi_U, g^*\Theta \rangle \in \Gamma(TU \otimes \mathfrak{g})$. For $r : U \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ and $\theta \in \Gamma(TU \otimes \mathfrak{g})$, we define

$$r^g := Ad_g \otimes Ad_g(r + \widehat{\langle g^*\Theta, \theta \rangle} + \langle \pi_U, g^*\Theta \wedge g^*\Theta \rangle), \quad (33)$$

$$\theta^g := Ad_g(\theta + \langle \pi_U, g^*\Theta \rangle), \quad (34)$$

where $\widehat{\langle g^*\Theta, \theta \rangle} : U \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ is the antisymmetrization of $\langle g^*\Theta, \theta \rangle$.

Proposition 2.22. *r^g is a generalized classical dynamical r -matrix coupled with U via θ^g with respect to Ω .*

Proof. Following Theorem 2.8, given the dynamical r -matrix (π_U, θ, r) , we can construct a right invariant bivector $\pi := \pi_U + \rho^L(\hat{\theta}) + \rho^L(r)$ on $U \times G$ such that $[\pi, \pi] = -2\rho(\Omega)$. We denote the graph of the map $g : U \rightarrow G$ by $U' \subset U \times G$. Then π is a right G invariant bivector fields and U' is a cross-section of the right G action. Following the argument in Theorem 2.7, associated to π and U' there exists a dynamical r -matrix $(\pi_{U'}, \theta_{U'}, r_{U'})$ with respect to Ω . To write down the explicit expressions of $\theta_{U'}$ and $r_{U'}$, we just need to compute the decomposition of $\pi|_{U'}$ with respect to the TU' and $\rho^R(\mathfrak{g})|_{U'}$.

We introduce an isomorphism $F : U \rightarrow U'$ by $F(x) = (x, g(x)) \in U'$ for any $x \in U$. A straightforward calculation gives that

$$F_*\pi_U = \pi_{U'}, \quad F^*\theta_{U'}^\sharp = \theta^g{}^\sharp \text{ and } r_{U'} \circ F = r^g.$$

It means that after identifying U' with U by F , the triple $(\pi_{U'}, \theta_{U'}, r_{U'})$ becomes (π_U, θ^g, r^g) . Thus r^g is automatically a generalized classical dynamical r -matrix coupled with (U, π_U) via θ^g with respect to Ω . It finishes the proof. ■

Definition 2.23. *We call the transformation $r \rightarrow r^g$, $\theta \rightarrow \theta^g$ the gauge transformation for (π_U, θ, r) with respect to $g : U \rightarrow G$.*

The geometric meaning of gauge transformations of generalized dynamical r -matrices is illuminated in the proof of Proposition 2.22. Another interpretation is given in the following: Let (M, π_M) be a (quasi-)Poisson G -manifold and (π_U, θ, r) be a dynamical r -matrix with respect to $\Omega = 0$ (the Cartan 3-tensor). Given a gauge transformation $g : U \rightarrow G$, we define a transformation from $U \times M$ to itself by

$$g \cdot (x, p) = (x, g(x) \cdot p), \quad \forall p \in G. \quad (35)$$

Proposition 2.24. *Let $(U \times M, \pi_r)$ and $(U \times M, \pi_{r^g})$ be the Poisson manifolds given in Theorem 2.12 associated to (π_U, θ, r) and (π_U, θ^g, r^g) respectively. Then we have*

$$\{F \circ g, G \circ g\}_r = \{F, G\}_{r^g} \circ g,$$

for any $F, G \in C^\infty(U \times G)$.

2.5 Generalized classical dynamical r -matrices and Poisson groupoids

In this subsection, we discuss the geometric interpretation of the generalized CDYB equation. Recall that in [7], Etingof and Varchenko found a geometric interpretation of the CDYB equation that generalizes Drinfeld's interpretation of the CYB equation in terms of Poisson-Lie groups. Namely, they constructed a so called dynamical Poisson-Lie groupoid structure on the direct product manifold $\eta^* \times G \times \eta^*$, where η is a Lie subalgebra of \mathfrak{g} . The CYBE can be viewed as the special case of the generalized CYBE (see Example 2.2). An observation here is that $\eta^* \times G \times \eta^*$ is the Lie groupoid integrating the Lie algebroid $T\eta^* \oplus \mathfrak{g}$. Furthermore, the Poisson structure on $\eta^* \times G \times \eta^*$ induces a Lie bialgebroid structure on $T\eta^* \oplus \mathfrak{g}$. Similarly, in the case of the generalized dynamical r -matrix, we have the following theorem. Let M be a manifold, \mathfrak{g} be a Lie algebra and $(TM \oplus \mathfrak{g}, [\cdot, \cdot]_L)$ be a Lie algebroid with the anchor map given by the projection to TM , the bracket given by

$$[X + A, Y + B]_L = [X, Y] + L_X B - L_Y A + [A, B]_{\mathfrak{g}}, \quad (36)$$

for all $X, Y \in \Gamma(TM)$ and $A, B \in C^\infty(M \times \mathfrak{g})$.

Theorem 2.25 (Theorem 4.5, [13]). *A solution r of the generalized DYBE coupled with (M, π) via θ induces a coboundary Lie bialgebroid structure $(TM \oplus \mathfrak{g}, d_*)$ where the differential $d_* : \Gamma(\wedge^\bullet(TM \oplus \mathfrak{g})) \rightarrow \Gamma(\wedge^{\bullet+1}(TM \oplus \mathfrak{g}))$ corresponding to the Lie algebroid structure on $T^*M \oplus \mathfrak{g}^*$ is of the form*

$$d_* = [\pi_M + \hat{\theta} + r, \cdot]_L,$$

where $[\cdot, \cdot]_L$ is the Schouten bracket on $\wedge^\bullet(TM \oplus \mathfrak{g})$.

A solution of the generalized Poisson-Lie DYBE coupled with (M, π_M) via θ also give a Lie bialgebroid $(TM \oplus \mathfrak{g}, d_*)$ where the differential $d_* = \delta + [\pi_M + \hat{\theta} + r, \cdot]$. According to the theory of integration of Lie bialgebroids in [16], we have

Corollary 2.26. *Associated to a generalized classical dynamical r -matrix coupled with (M, π) via θ , there is a Poisson Lie groupoid structure on $\mathcal{G} = M \times G \times M$ whose tangent Lie bialgebroid is $(TM \oplus \mathfrak{g}, d_*)$.*

Thus the Poisson Lie groupoid $M \times G \times M$ gives a geometric interpretation of the generalized DYBE that generalizes Drinfeld's interpretation of the CYBE in terms of Poisson-Lie groups.

A smooth manifold M is called \mathcal{G} -space for a Lie groupoid $(\mathcal{G} \rightrightarrows P, s, t)$ if there are two smooth maps, the moment map and the action map, $J : M \rightarrow P$ and

$$\alpha : \mathcal{G} \times_J M = \{(x, m) \in \mathcal{G} \times M \mid t(x) = J(m)\} \rightarrow M$$

such that, writing $\alpha(x, m) = x \cdot m$, for all compatible $x, y \in \mathcal{G}$ and $m \in M$,

- (i) $J(x \cdot m) = s(x)$;
- (ii) $(x \cdot y) \cdot m = x \cdot (y \cdot m)$;
- (iii) $J(m) \cdot m = m$.

Now suppose that M is a \mathcal{G} -space. The action of \mathcal{G} on M is a Poisson action if its graph $\{(x, m, x \cdot m) \mid t(x) = J(m)\}$ is a coisotropic submanifold of $\mathcal{G} \times M \times \overline{M}$ [17]. Then M is called a Poisson \mathcal{G} -space.

By Theorem 2.9 and Corollary 2.26, associated to a dynamical r -matrix r coupled with (M, π_M) via θ with respect to the Cartan 3-tensor and a quasi-Poisson G -space N , we have a Poisson Lie groupoid $\mathcal{G} = M \times G \times M$ and a Poisson manifold $(M \times N, \pi = \pi_M + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)$. Further, there is a natural action of the groupoid $\mathcal{G} \rightrightarrows M$ on $M \times N$ defined by

$$(x, g, y) \cdot (y, p) = (x, g \cdot p)$$

for all $x, y \in M$, $g \in G$ and $p \in N$. This is a Lie groupoid action with respect to the moment map $J : M \times N \rightarrow M$ given by the natural projection. An observation here is that this action is a Poisson action.

Theorem 2.27. *$(M \times N, \pi)$ is a Poisson \mathcal{G} -space.*

To prove this theorem, we need the following results.

Lemma 2.28 (Theorem 3.3, [12]). *Let \mathcal{G} be a Poisson groupoid with its tangent Lie bialgebroid (A, A^*) . Then a Poisson manifold (M, π) is a Poisson \mathcal{G} -space if and only if the vector bundle morphism from T^*M to A^* , the dual of the infinitesimal action map, is a Lie algebroid morphism.*

Lemma 2.29 (Theorem 3.1, [12]). *Let A^* be a Lie algebroid over P , and (M, π) a Poisson manifold. Then, for the cotangent Lie algebroid T^*M induced by the Poisson structure, a vector bundle morphism $\Phi : T^*M \rightarrow A^*$ over $J : M \rightarrow P$ is a Lie algebroid morphism if and only if the following two conditions hold:*

- (i) $H_{J^*f} = -\Phi^*(d_*f)$, $\forall f \in C^\infty(P)$;
- (ii) $L_{\Phi^*(X)}\pi = -\Phi^*(d_*S)$, $\forall S \in \Gamma(A)$,

where H_{J^*f} denotes the Hamiltonian vector field on M , which is defined by $H_{gh} = \pi(dg, dh)$ for all $g, h \in C^\infty(M)$. The differential d_* comes from the Lie algebroid structure on A^* .

Proof of Theorem 2.27 In our case, the Poisson manifold is $(M \times N, \pi = \pi_M + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)$ and the Lie bialgebroid is $(TM \otimes \mathfrak{g}, d_* = [\pi_M + \hat{\theta} + r, \cdot])$. So by Lemma 2.28 and Lemma 2.29, we just need to prove $H_{J^*f} = -F(d_*f)$, $\forall f \in C^\infty(M)$ and $L_{F(S)}\pi_M = -F(d_*S)$, $\forall S \in \Gamma(TM \otimes \mathfrak{g})$, where the bundle map $F : TM \oplus \mathfrak{g} \rightarrow T(M \times N)$ is the infinitesimal action of $M \times G \times M$ on $M \times N$, explicitly given by $F(X + e) = X + \rho(e)$ for $X \in \Gamma(TM)$ and $e \in \mathfrak{g}$.

(1) Following the expression of the Poisson tensor π on $M \times N$ given in Theorem 2.9, we have for all $f \in C^\infty(M)$,

$$H_{J^*f} = \pi_M^*(df) + \rho(\theta^*(df)),$$

where $J : M \times N \rightarrow M$ is the natural projection map. On the other hand, by the definition of the differential d_* ,

$$d_*f = [\pi_M, f] + [\hat{\theta}, f] + [r, f].$$

Note that $[r, f] = 0$, $F([\pi_M, f]) = [\pi_M, f]$, and $F([\hat{\theta}, f]) = F(-\theta^*(df)) = -\rho(\theta^*(df))$. Therefore,

$$H_{J^*f} = -[\pi_M + \rho(\hat{\theta}) + \rho(r), f] = -F(d_*f).$$

(2) We assume $S = X + e \in \Gamma(TM \oplus \mathfrak{g})$. Then

$$d_*S = [\pi_M + \hat{\theta} + r, X + e] = [\pi_M + \hat{\theta} + r, X] + [\pi_M + \hat{\theta} + r, e].$$

By the definition of $F : \wedge^*(TM \oplus \mathfrak{g}) \rightarrow T(M \times N)$, we have that the map F and the Schouten-bracket $[\cdot, \cdot]_L$ on $\wedge^*(TM \otimes \mathfrak{g})$ are commutative. As a result,

$$F([\pi_M + \hat{\theta} + r, X]) = [F(\pi_M) + F(\hat{\theta}) + F(r), F(X)] = [\pi, X].$$

Similarly,

$$F([\pi_M + \hat{\theta} + r, e]) = [F(\pi_M) + F(\hat{\theta}) + F(r), F(e)] = [\pi, \rho(e)].$$

Eventually, we get $-F(d_*S) = L_{F(S)}\pi$. This finishes the proof. ■

3 Generalized dynamical r -matrices and moduli spaces of flat connections on surfaces

3.1 Poisson and quasi-Poisson structures on the moduli spaces of flat connections on surfaces

In [3], Atiyah-Bott introduced canonical symplectic structures on the moduli spaces of flat connections over oriented surfaces. Let $\Sigma_{g,n}$ be an oriented surface of genus g with n punctures. A convenient description of the moduli space is given by the character variety, i.e., the space of group homomorphisms $h : \pi_1(\Sigma_{g,n}) \rightarrow G$ that map the homotopy equivalence class of a loop around the i -th puncture to an associated conjugacy class \mathcal{C}_i . Two such group homomorphisms describe gauge-equivalent connections if and only if they

are related by conjugation with an element of G . This implies that the moduli space of flat G -connections on $\Sigma_{g,n}$ is given by

$$X_G(\Sigma_{g,n}) = \text{Hom}_{\mathcal{C}_1, \dots, \mathcal{C}_n}(\pi_1(\Sigma_{g,n}, G))/G = \{h \in \text{Hom}(\pi_1(\Sigma_{g,n}), G) \mid h(m_i) \in \mathcal{C}_i\}/G,$$

where $m_i \in \pi_1(\Sigma_{g,n})$ corresponds to the loop around i -th puncture. By characterising the group homomorphisms in terms of the images of the generators of $\pi_1(\Sigma_{g,n})$, the moduli space of flat connections is the set

$$\{(M_1, \dots, M_n, A_1, B_1, \dots, A_g, B_g) \in G^{n+2g} \mid M_i \in \mathcal{C}_i, [B_g, A_g] \cdots [B_1, A_1] \cdot M_n \cdots M_1 = 1\}/G,$$

where the quotient is taken with respect to the diagonal action of G on G^{n+2g} . This moduli space carries a canonical symplectic structure which is obtained via reduction from the canonical symplectic structure on the space of connections on $\Sigma_{n,g}$. Following Fock-Rosly's approach, an explicit description of this symplectic structure is obtained by Poisson reduction of a Poisson structure on an enlarged ambient space G^{n+2g} . In this description, the i -th component of G^{n+2g} corresponds to the holonomy along a generator of the fundamental group $\pi_1(\Sigma_{g,n})$. Thus the space of representations is embedded into G^{n+2g} . There are two natural operators $\nabla_R, \nabla_L \in \Gamma(TG \otimes \mathfrak{g}^*)$ given for all $A \in \mathfrak{g}, p \in G$ by

$$\begin{aligned} \langle \nabla_R, A \rangle f(p) &:= \frac{d}{dt} \Big|_{t=0} f(p \cdot \exp(-tA)), \\ \langle \nabla_L, A \rangle f(p) &:= \frac{d}{dt} \Big|_{t=0} f(\exp(tA) \cdot p). \end{aligned}$$

With them, we define $2(n+2g)$ covariant differential operators in the following way:

$$\begin{aligned} \nabla_{2i-1} &= \nabla_R^{M_i}, & \nabla_{2i} &= \nabla_L^{M_i} \text{ for } i = 1, \dots, n; \\ \nabla_{n+4i-3} &= \nabla_R^{A_i}, & \nabla_{n+4i-1} &= \nabla_L^{A_i} \text{ for } i = 1, \dots, g; \\ \nabla_{n+4i-2} &= \nabla_R^{B_i}, & \nabla_{n+4i-1} &= \nabla_L^{B_i} \text{ for } i = 1, \dots, g. \end{aligned} \tag{37}$$

Definition 3.1. Let G be a Lie group with Lie algebra \mathfrak{g} . For any $r \in \mathfrak{g} \otimes \mathfrak{g}$, the corresponding Fock and Rosly's bivector $B_r^{n,g} \in \Gamma(\wedge^2(TG^{n+2g}))$ is defined by

$$B_r^{n,g}(df, dh) := \frac{1}{2} \sum_i \langle r, \nabla_i f \wedge \nabla_i h \rangle + \sum_{i < j} \langle r, \nabla_i f \wedge \nabla_j h \rangle. \tag{38}$$

Theorem 3.2. [10] Let \mathfrak{g} be a Lie algebra with a non-degenerate Ad-invariant symmetric bilinear form $K(\cdot, \cdot)$. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

then $B_r^{n,g}$ defines a Poisson structure on G^{n+2g} . Furthermore, when the symmetric part of r coincides with the Casimir element κ , this Poisson structure induces the canonical symplectic structure on the moduli space of flat G -connections on $\Sigma_{g,n}$.

From the expression of $B_r^{n,g}$, we see that the Poisson bracket of two functions on G^{n+2g} depends only on the symmetric component of r if one of the two functions is invariant under the diagonal action of G on G^{n+2g} . So we can use the symmetric part of r and reduction procedure to describe the Poisson structure on the quotient space G^{n+2g}/G . In this case instead of the Poisson structure, we have a quasi-Poisson structure on G^{n+2g} . Actually, a main result in quasi-Poisson theory is that the standard Poisson structure on $X_G(\Sigma_{n,g})$ is given by a reduction of a canonical quasi-Poisson tensor on $M_G(\Sigma_{n,g}) = \{h \in \text{Hom}(\pi_1(S_{g,n}, G) | h(m_i) \in \mathcal{C}_i)\}$.

Theorem 3.3. [1] *Consider the quasi-Poisson manifold*

$$P_{g,n} = \mathcal{C}_1 * \dots * \mathcal{C}_n * D(G) * \dots * D(G),$$

where $\mathcal{C}_1, \dots, \mathcal{C}_n$ are conjugacy classes in G . Then the quasi-Poisson reductions of $P_{g,n}$ are isomorphic to the moduli spaces of flat connections on $\Sigma_{g,n}$ with the Atiyah-Bott symplectic form.

A direct computation shows that the quasi-Poisson tensor on $P_{g,n}$ coincides with the restriction of $B_\kappa^{g,n}$ to the set of conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_n$, where $B_\kappa^{g,n}$ is the bivector field given in (38) with respect to the Casimir element $\kappa \in S^2\mathfrak{g}$.

3.2 GCDYB equations and moduli spaces of flat connections on surfaces

In this subsection, we will combine the discussion in previous sections and give our main result which describes the canonical symplectic structure on the moduli spaces of flat connections on surfaces by using generalized dynamical r -matrices.

Following Theorem 3.3, the symplectic structure on $X_G(\Sigma_{g,n})$ is given by the reduction of the quasi-Poisson structure $B_\kappa^{g,n}$ on $P_{g,n} = \mathcal{C}_1 * \dots * \mathcal{C}_n * D(G) * \dots * D(G)$ with respect to the simultaneous conjugation action of G .

Reduction with respect to two punctures

We assume that there are more than two punctures on the surface $\Sigma_{g,n}$. We choose a cross-section of the G action on $\Sigma_{g,n}$. Then the reduced Poisson structure on it is viewed to be a local model of the Poisson structure on $X_G(\Sigma_{g,n})$. We do the reduction in a “minimal” way, i.e., imposing gauge fixing conditions on the first two punctures in the following way. We see $P_{g,n}$ as the fusion product of $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_{G^2})$ and $(P_{g,n-2}, B_\kappa^{g,n-2})$, i.e., $P_{g,n} = (\mathcal{C}_1 * \mathcal{C}_2) * P_{g,n-2}$ in which $P_{g,n-2} := \mathcal{C}_3 * \dots * \mathcal{C}_n * D(G) * \dots * D(G)$. Let U be any cross-section of the diagonal action of G on $\mathcal{C}_1 \times \mathcal{C}_2$ and (π_U, θ, r) the moduli space generalized dynamical r -matrix associated to U . By Theorem 2.9, we obtain the reduced Poisson structure on $U \times P_{g,n-2}$:

$$\pi_{\text{red}} = \pi_U + \rho(\hat{\theta}) + \rho(r) + B_\kappa^{g,n-2},$$

where $\rho : \mathfrak{g} \rightarrow P_{g,n-2}$ is the infinitesimal action generated by simultaneous conjugation G action. Moreover a simple comparison gives that $\rho(r) = B_r^{g,n-2}$ as bivector fields on

$U \times P_{g,n-2}$, where $B_r^{g,n-2}$ is given by (38) for $r : U \rightarrow \mathfrak{g} \wedge \mathfrak{g}$. As a result, $\rho(r) + B_\kappa^{g,n-2} = B_{r+\kappa}^{g,n-2}$, where r and κ can be seen as antisymmetric and symmetric parts of the function $r + k \in C^\infty(U, \mathfrak{g} \otimes \mathfrak{g})$. Eventually, as an application of Theorem 2.12, we get

Theorem 3.4. *The quasi-Poisson structure $B_\kappa^{g,n}$ on $P_{g,n}$ induces a Poisson bracket on $U \times \mathcal{C}_3 \dots \times \mathcal{C}_n \times G^{2g}$, which is isomorphic to the Atiyah-Bott symplectic structure and takes the following form:*

(1) For $f, g \in C^\infty(\mathcal{C}_3 \dots \times \mathcal{C}_n \times G^{2g})$.

$$\{f, g\} = B_{r+\kappa}^{n-2,g}(df, dg) \quad (39)$$

(2) For $f \in C^\infty(\mathcal{C}_3 \dots \times \mathcal{C}_n \times G^{2g})$ and $\phi, \varphi \in C^\infty(U)$:

$$\{f, \phi\} = \rho(\hat{\theta})(df, d\phi) \quad (40)$$

$$\{\phi, \varphi\} = \pi_U(d\phi, d\varphi), \quad (41)$$

Note that the original Fock and Rosly's bivector $B_r^{n,g}$ on G^{n+2g} is related to a classical r -matrix which is a constant element in $\mathfrak{g} \otimes \mathfrak{g}$. Here we describe the Poisson tensor on the quotient space by a dynamical r -matrix which is a map from U to $\mathfrak{g} \wedge \mathfrak{g}$.

Corollary 3.5. *Let \mathcal{G} be the Poisson Lie groupoid associated to the moduli space dynamical r -matrix (π_U, r, θ) , then $(U \times \mathcal{C}_3 \dots \times \mathcal{C}_n \times G^{2g}, \{\cdot, \cdot\})$ carries a Poisson \mathcal{G} action.*

Gauge fixing and classical dynamical r -matrices in ISO(2,1)-Chern-Simons theory.

In [15], Meusburger and Schönfeld obtained classical dynamical r -matrices by considering gauge fixing in ISO(2,1)-Chern-Simons theory. Now, we interpret these classical dynamical r -matrices as moduli space dynamical r -matrices corresponding to $G = \text{ISO}(2, 1)$.

First, let us give the required notations. We denote by $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$ the standard basis of \mathbb{R}^3 . By ε_{abc} we denote the totally antisymmetric tensor in three dimensions with the convention $\varepsilon_{012} = 1$. The indices of ε_{abc} are raised with the three-dimensional Minkowski metric $\eta = \text{diag}(1, -1, -1)$.

The Poincaré group in 3-D is the semidirect product $\text{ISO}(2, 1) = \text{SO}_+(2, 1) \ltimes \mathbb{R}^3$ of the proper orthochronous Lorentz group $\text{SO}_+(2, 1)$ and the translation group \mathbb{R}^3 . The elements of $\text{ISO}(2, 1)$ are parameterized as

$$(u, \mathbf{a}) = (u, 0) \cdot (1, -\mathbf{j}) = (u, -\text{Ad}(u)\mathbf{j}) \text{ with } u \in \text{SO}_+(2, 1), \mathbf{j}, \mathbf{a} \in \mathbb{R}^3.$$

The corresponding coordinate functions $\{j^a\}_{a=0,1,2}$ are given by

$$j^a : \text{ISO}(2, 1) \rightarrow \mathbb{R}, \quad (u, -\text{Ad}(u)\mathbf{q}) \rightarrow q^a.$$

Let $\{J_a\}_{a=0,1,2}$ be a basis of $\mathfrak{so}(2, 1)$ such that the Lie bracket takes the form $[J_a, J_b] = \varepsilon_{ab}^c J_c$. Hence a basis of the Lie algebra $\mathfrak{iso}(2, 1)$ is given by $\{J_a\}_{a=0,1,2}$ together with a basis $\{P_a\}_{a=0,1,2}$ of the abelian Lie algebra \mathbb{R}^3 .

The moduli space of flat G -connections can be viewed as a constrained system in the sense of Dirac [4]. From the expression for the moduli space of flat connections, it is apparent that the moduli space is obtained from $P_{g,n}$ by imposing a group-valued constraint that arises from the defining relation of the fundamental group $\pi_1(\Sigma_{g,n})$.

In the case of $G = \text{ISO}(2,1)$, the group-valued constraint from the defining relation of the fundamental group $\pi_1(\Sigma_{g,n})$ can be viewed as a set of six first constraints in the Dirac gauge fixing formalism for the Fock-Rosly bracket. The associated gauge transformations which they generate via the Poisson bracket are given by the diagonal action of $\text{ISO}(2,1)$ on $\text{ISO}(2,1)^{n+2g}$.

A choice of gauge fixing conditions for the constraints is investigated in [15]. These gauge fixing conditions implement the quotient by $\text{ISO}(2,1)$ and restrict the first two components of all points $(M_1, \dots, B_g) \in \Sigma = C^{-1}(0)$ in such a way that M_1, M_2 are determined uniquely by two real parameters ψ and α given in terms of the components of the product $M_2 \cdot M_1 = (u_{12}, -\text{Ad}(u_{12})\mathbf{j}_{12})$ as

$$\psi = f(\text{Tr}(u_{12})), \quad \alpha = g(\text{Tr}(u_{12}))\text{Tr}(j_{12}^a J_a \cdot u_{12}) + h(\text{Tr}(u_{12})), \quad (42)$$

where $f, g \in C^\infty(\mathbb{R})$ are arbitrary diffeomorphisms and $h \in C^\infty(R)$. This allows us to identify the constraint surface $\Sigma = C^{-1}(0)$ with a subset of $\mathbb{R}^2 \times \text{ISO}(2,1)^{n-2+2g}$, where the \mathbb{R}^2 is parameterized by (ψ, α) and $\text{ISO}(2,1)^{n-2+2g}$ by (M_3, \dots, B_g) .

We choose a set of constraint functions of such form that satisfy the requirements of Dirac gauge fixing. Following Theorem 4.5 in [15], there exist maps $\mathbf{q}_\psi, \mathbf{q}_\alpha, \mathbf{q}_\delta, m : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $V : \mathbb{R} \rightarrow \text{Mat}(3, \mathbb{R})$ such that the associated Dirac bracket is given in terms of them. On the other hand, the Dirac gauge fixing corresponds to the constraint functions is equivalent to choose a cross-section of the $\text{ISO}(2,1)$ action on $\mathcal{C}_1 \times \mathcal{C}_2$ which is the locus of the six functions $\mathcal{C}_{i=1, \dots, 6}$. Thus by Theorem 3.4, there is a moduli space dynamical r -matrix corresponding to this cross-section. We give the following theorem which interprets the main result in [15] in our framework.

Theorem 3.6. *The moduli space dynamical classical r -matrix (π, θ, r) corresponding to the Dirac gauge fixing procedure is given by*

$$\pi = 0, \quad \theta = q_\alpha^a \frac{\partial}{\partial \alpha} \otimes J_a + q_\psi^a \frac{\partial}{\partial \psi} \otimes P_a + q_\delta^a \frac{\partial}{\partial \alpha} \otimes P_a, \quad (43)$$

$$r = -V^{bc}(\psi)(P_b \otimes J^c - J^c \otimes P_b) + \varepsilon^{bcd} m_d(\psi, \alpha) P_b \otimes P_c. \quad (44)$$

Moreover, the induced Poisson bracket takes the following form:
for any $f, g \in C^\infty(\text{ISO}(2,1)^{n-2+2g})$,

$$\{\alpha, \psi\} = 0, \quad \{\alpha, f\} = \rho(\hat{\theta})(d\alpha, df), \quad \{f, g\} = B_{r+\kappa}^{n-2,g}(df, dg) \quad (45)$$

where $\kappa = P_a \otimes J^a$.

Given a map $p : \mathbb{R}^2 \rightarrow \text{ISO}(2, 1)$, let us consider the smooth map

$$\begin{aligned}\Phi^p : \mathbb{R}^2 \times \text{ISO}(2, 1)^{n-2+2g} &\rightarrow \mathbb{R}^2 \times \text{ISO}(2, 1)^{n-2+2g}, \\ (\psi, \alpha, M_3, \dots, B_g) &\rightarrow (\psi, \alpha, \text{Ad}_p M_3, \dots, \text{Ad}_p B_g).\end{aligned}$$

As a consequence of Proposition 3.7, we have

Corollary 3.7. *Let $\{\cdot, \cdot\}$ be the bracket given in (45) with respect to (θ, r) . Then for all $F, G \in C^\infty(\mathbb{R}^2 \times \text{ISO}(2, 1)^{n-2+2g})$,*

$$\{F \circ \Phi^p, G \circ \Phi^p\} = \{F, G\}^p \circ \Phi^p, \quad (46)$$

where $\{\cdot, \cdot\}_D^p$ is the bracket given in (45) with respect to (θ^p, r^p) , the gauge transformation of (θ, r) via $p : \mathbb{R}^2 \rightarrow \text{ISO}(2, 1)$.

Particularly, the map $p = (g, -\text{Ad}(g)\mathbf{t})$ satisfying $\partial_\alpha g = \partial_\alpha^2 \mathbf{t} = 0$ is called dynamical Poincaré transformation in [15]. Actually, dynamical r -matrices from different gauge fixing conditions subject to extra conditions given in [15] are related by dynamical Poincaré transformations. In this special case, Corollary 3.7 turns to be Lemma 5.1 in [15].

A simplified standard set of dynamical r -matrices from the Dirac gauge fixing in $\text{ISO}(2, 1)$ -Chern-Simons theory is given explicitly in [15]. This set of solutions corresponds to special gauge fixing condition which is motivated by its direct physical interpretation in the application to the Chern-Simons formulation of $(2 + 1)$ -gravity.

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